Reduced Row Echelon Form

Here are the steps involved in converting a matrix to reduced rear echelon form.

The Goal

We want to convert the augmented matrix for a system of equations to a new augmented matrix whose coefficient matrix part is the unit matrix; *i.e.*, convert

$$\left[\begin{array}{ccc|c} a & b & c & m \\ d & e & f & n \\ g & h & i & p \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$$

Step 1: Shuffle and reduce

- Any row that already has a zero in the left column should be moved to the second or third row
- Any row with zeros in both the first two columns should be moved to the bottom row.
- Any row that can be reduced should be so.

Step 2: Use row operations to change the values in the lower-left "L" of the matrix to zeros.

$$\left[\begin{array}{ccc|c} a & b & c & m \\ d & e & f & n \\ g & h & i & p \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} a & b & c & q \\ 0 & j & k & r \\ 0 & 0 & r & s \end{array}\right]$$

Step 3: Use row operations to change the values in the upper-right "L" of the matrix to zeros.

$$\left[\begin{array}{ccc|c} a & b & c & q \\ 0 & j & k & r \\ 0 & 0 & r & s \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} s & 0 & 0 & t \\ 0 & j & 0 & u \\ 0 & 0 & r & v \end{array}\right]$$

Step 4: Divide each row by the number needed to change the row's value to 1.

$$\left[\begin{array}{ccc|c} s & 0 & 0 & t \\ 0 & j & 0 & u \\ 0 & 0 & r & v \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$$

For Example

Solve: $\left[\begin{array}{ccc|c} -2 & 2 & 4 & 2 \\ 2 & 3 & 1 & -2 \\ 5 & 4 & 2 & 4 \end{array}\right]$

Step 1: Shuffle and reduce

$\frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ 5 & 4 & 2 & 4 \end{array}\right]$$

Step 2: Change lower-right L to zeros

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 0 & 5 & 5 & 0 \\ 0 & 9 & 12 & 9 \end{array}\right]$$

$2R_1 + R_2$

$5R_1 + R_3$

Let's reduce Line 2

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & 12 & 9 \end{array}\right]$$

$\frac{1}{5}R_2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$9R_2 - R_3$

Step 3: Change upper-right L to zeros

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 9 \end{array}\right]$$

$R_1 - R_2$

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 12 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 9 \end{array}\right]$$

$3R_1 + R_3$

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 9 \end{array}\right]$$

$3R_2 + R_3$

Step 4: Divide to make elements = 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$-\frac{1}{3}R_1$

$-\frac{1}{3}R_3$