Tangent and Normal Vectors

Tangent Vectors

If $\mathbf{r}(t)$ is a smooth curve, the *unit tangent vector* at t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \qquad \mathbf{r}'(t) \neq \mathbf{0}$$

- The *tangent line to a curve* at a point is the line parallel to **T** that passes through that point.

Smoothness

A curve $\mathbf{r}(t)$ is smooth on an interval when $\mathbf{r}'$ is continuous and non-zero in the interval.

Principal Unit Normal Vector

If **r** is a smooth curve and $\mathbf{T}'(t) \neq \mathbf{0}$, the *principal unit normal vector* at t is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \qquad \mathbf{T}'(t) \neq \mathbf{0}$$

- **N** is orthogonal to **T** at t and points to the concave side of the curve.

Plane Curves

If a plane curve has unit tangent vector

$$\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

the principal unit normal vector must be either

$$\mathbf{N}_1(t) = y(t)\mathbf{i} - x(t)\mathbf{j} \quad \text{or} \quad \mathbf{N}_2(t) = -y(t)\mathbf{i} + x(t)\mathbf{j}$$

The *principal* unit normal vector is the one that points toward the concave side of the curve.

Motion

If $\mathbf{r}$(t) is a position vector for a curve C for which $\mathbf{N}$(t) exists, then the tangential and normal components of the acceleration are:

Tangential Component of acceleration

$$a_{\mathbf{T}} = \frac{d}{dt}\left[\|\mathbf{v}\|\right] = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

Normal Component of acceleration

$$a_{\mathbf{N}} = \|\mathbf{v}\|\,\|\mathbf{T}'\| = \mathbf{a}\cdot\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 + a_{\mathbf{T}}^2}$$