Arc Length and Curvature

Arc Length, *s*

If C is a smooth curve defined by the vector-valued equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then the length, s, along the curve from $t = a$ to $t = b$ is:

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Arc Length Parameter, *s*(*t*)

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du$$

Why Convert r(*t*) to r(*s*)?

It is often useful to rewrite a curve's equation as $\mathbf{r}(s)$ because of the following theorem:

$$\|\mathbf{r}'(s)\| = 1$$

Curvature, *K*

Let C be a smooth curve defined by the vector-valued equations $\mathbf{r}(t)$ or $\mathbf{r}(s)$,

If you have r(s)...

$$K = \left\|\frac{d\mathbf{T}}{ds}\right\| = \|\mathbf{T}'(s)\|$$

The tangent vector, $\mathbf{T} = \dfrac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|}$

If you have r(t)...

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{a}(t) \cdot \mathbf{N}(t)\|}{\|\mathbf{v}(t)\|^2}$$

$\mathbf{a}(t)$ = acceleration; normal $\mathbf{N}(t) = \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$

If you have a planar figure, y = f(x)...

$$K = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}}$$

Radius of Curvature, *R*

$$R = \frac{1}{K}$$

If you have a planar figure, <x(t), y(t)>...

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{\frac{3}{2}}}$$

Acceleration, Speed, and Curvature

If $\mathbf{r}(t)$ is the position vector for a particle's motion along a smooth curve, C, then the acceleration vector is

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}$$