Finding Multivariable Extrema

Finding Extrema Using Critical Numbers

Determining Critical Numbers

(x_0, y_0) is a critical number of $f(x,y)$ if one of the two conditions are true:

- $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$
- $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ does not exist

Second Partials Test

If $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$, then (x_0,y_0) is a local maximum, local minimum, or a saddle point. We can determine which it is using the *Second Partials Test*.

Second Partials Test

To test whether a critical point (x_0,y_0) is a local max, min, or saddle point:

1. If $f_x(x_0,y_0) \neq 0$ or $f_y(x_0,y_0) \neq 0$, then the point is not a max, min, or saddle point; you're done.
2. Otherwise, calculate a value, *d:*

$$d = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - [f_{xy}(x_0,y_0)]^2$$

3. Interpret *d* as follows:
   - If $d > 0$ and $f_{xx}(x_0,y_0) > 0$, then (x_0,y_0) is a relative **minimum**
   - If $d > 0$ and $f_{xx}(x_0,y_0) < 0$, then (x_0,y_0) is a relative **maximum**
   - If $d < 0$, then $(x_0,y_0, f_x(x_0,y_0))$ is a relative **saddle point**
   - If $d = 0$, the test is inconclusive

Lagrange Multipliers on next page

Lagrange Multipliers

Lagrange multipliers allow you to find the maximum and minimum values of a multivariable function $f(x,y)$ in the presence of constraints. The presentation here is for two variables, x and y, but can be easily extended to more.

Notation

$f(x,y)$ is the function

$g(x,y) = c$ is the constraint

We shall introduce a new variable, λ (our Lagrange multiplier), such that

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

You do remember ∇, yes?

The gradient, ∇, is a vector quantity defined as:

$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$

Extend the pattern for more variables:

$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$

The Method

1 Solve the system of equations:

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = c$$

Usually, you will solve one of the equations for λ, then substitute that into the next equation, etc.

2 Evaluate $f(x,y)$ at each of the solution points found in step 1.

- The greatest value will be the maximum.
- The least value will be the minimum.