Tests for Series Convergence

Geometric Series

Given a geometric series of the form $\sum_{n=0}^{\infty} ar^n$, the series converges if $-1 < r < 1$.

Nth-Term Test

Given $\sum_{n=1}^{\infty} a_n$ and $L = \lim_{n\to\infty} a_n$

- If L = 0, the series converges
- If L ≠ 0, the series diverges

Important note: the Nth-term test cannot reliably be used to test for convergence.

p-Series

Given a p-series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$

The series converges if the constant $p > 1$.

Harmonic Series

A *harmonic series* is a p-series in which p = 1:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + ½ + ⅓ + \ldots$$

A *general harmonic series* is of the form

$$\sum_{n=1}^{\infty} \frac{1}{an+b}$$

Alternating Series

Given alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ and $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$

where $a_n > 0$

The series converge when two conditions are met:

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ for all *n*. (*i.e.*, the series is monotonically decreasing.)

Absolute convergence

If $\sum|a_n|$ converges, then $\sum a_n$ also converges. (Note the converse is not true.)

Absolute vs Conditional

Given convergent series Σa_n

- Σa_n is *absolutely convergent* if $\Sigma|a_n|$ is also convergent.
- Σa_n is *conditionally convergent* if $\Sigma|a_n|$ is divergent.

Remainder (maximum error)

Let S_N be the Nth partial sum of an alternating series; the maximum difference, R_N (also called the maximum error or remainder), between S_N and the sum of the entire series is

$$|R_N| \le a_{N+1}$$

∞

Telescoping Series

Given telescoping series $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \dots + (b_n - b_{n+1})$

- Series converges if $\lim_{n\to\infty} b_{n+1}$ is finite
- Further, the sum of the series, $S = b_1 - \lim_{n\to\infty} b_{n+1}$

Integral Test

Given $a_n = f(n)$, where $f(n)$ is positive, continuous, and decreasing for all n,

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} a_n$ are either both convergent or both divergent.

Remainder for Integral Test

If a_k is countinuous, positive, and decreasing for $x \geq$ some value n, and $\sum a_n$ is convergent, and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

$n+1$

Root Test

Given $\sum a_n$ and $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- If $L < 1$, the series converges
- If $L > 1$, the series diverges
- If $L = 1$, the test is inconclusive

Ratio Test

Given $\sum a_n$ where a_n is non-zero for all n and $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

- If $L < 1$, the series converges absolutely
- If $L > 1$ or $L = \infty$, the series diverges
- If $L = 1$, the test is inconclusive

Comparison Test (Direct Comparison Test)

Given $0 \leq a_n \leq b_n$ for all values of n,

- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges
- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges

Using the Comparison Tests

We often use the comparison and limit comparison tests to compare a series with a p-series or geometric series.

Limit Comparison Test

Given $a_n, b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$

If L is finite and positive, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.