Taylor and Maclaurin Series

What are they?

- A *Taylor series* generates a polynomial function that approximates a function $f(x)$ for values of x near a value c for which that function's value is known.

  e.g., you could use a Taylor polynomial to find the sines of angles near $\frac{\pi}{2}$, given that we know $\sin(\frac{\pi}{2}) = 1$. In this case, $c = \frac{\pi}{2}$ and we say that this Taylor series is *centered* on $\frac{\pi}{2}$.

  - A polynomial derived by simplifying the first n elements of a Taylor series is referred to as an *nth-degree Taylor polynomial.*

- A *Maclaurin series* is a Taylor series centered on 0, that is, $c = 0$.

Calculation

Taylor Series

Note that the coefficients here are 1/0!, 1/1!, 1/2!, 1/3!, etc.

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

That is,

$$F(x) = f(c) + f'(c)(x-c) + \tfrac{1}{2}f''(c)(x-c)^2 + \tfrac{1}{6}f'''(c)(x-c)^3 + \ldots$$

Maclaurin Series

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x^n)$$

That is,

$$F(x) = f(0) + f'(0)(x) + \tfrac{1}{2}f''(0)(x^2) + \tfrac{1}{6}f'''(0)(x^3) + \ldots$$

Accuracy

The accuracy of an nth-degree Taylor or Maclaurin series to the actual function at a given value of x is built around the notion of the *remainder*, $R_n(x)$, which is the difference between the value of the series and the actual value of the function. The maximum error in the series at a given value of x, compared to the actual value of the target function, is:

$$|R_n(x)| \leq \frac{|x-c|^{n+1}}{(n+1)!} \max |f^{(n+1)}(z)|$$

The phrase "$\max |f^{(n+1)}(z)|$" refers to the maximim value that the derivatives of $f(x)$ can have for any value z between x and c. For example, if the function is *sin(x)*, then the maximum value would be 1.