Taylor and Maclaurin Series

What are they?

- A *Taylor series* generates a polynomial function that approximates a function $f(x)$ for values of x near a value c for which that function's value is known.

  e.g., you could use a Taylor polynomial to find the sines of angles near $\frac{\pi}{2}$, given that we know $\sin(\frac{\pi}{2}) = 1$. In this case, $c = \frac{\pi}{2}$ and we say that this Taylor series is *centered* on $\frac{\pi}{2}$.

  ▹ The polynomial derived by simplifying the first n elements of a Taylor series is referred to as an *nth-degree Taylor polynomial.*

- A *Maclaurin series* is a Taylor series centered on 0, that is, $c = 0$.

Calculation

Taylor Series

Note that the coefficients here are 1/0!, 1/1!, 1/2!, 1/3!, etc.

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

That is,

$$F(x) = f(c) + f'(c)(x-c) + \tfrac{1}{2}f''(c)(x-c)^2 + \tfrac{1}{6}f'''(c)(x-c)^3 + \ldots$$

Maclaurin Series (Taylor series centered on 0)

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x^n)$$

That is,

$$F(x) = f(0) + f'(0)(x) + \tfrac{1}{2}f''(0)(x^2) + \tfrac{1}{6}f'''(0)(x^3) + \ldots$$

Accuracy: Remainder

The remainder is also called the Lagrange error bound.

The *accuracy* of an nth-degree Taylor or Maclaurin series at a given value of x is built around the notion of the *remainder*, $R_n(x)$, which is the difference between the value of the series and the actual value of the function at that value of x. The *maximum error* in the series at a given value of x is:

$$|R_n(x)| \leq \frac{|x-c|^{n+1}}{(n+1)!} \max |f^{(n+1)}(z)|$$

The phrase "max $|f^{(n+1)}(z)|$" refers to the maximim absolute value that the derivatives of $f(x)$ can have for any value z between x and c. For example, if the function is *sin(x)*, then the maximum absolute value would be 1 (because $\sin(x)$ varies between –1 and 1).

Common Taylor Series

$$\frac{1}{1-\mathrm{x}} = \sum_{n=0}^{\infty} x^n = 1 + x + x^3 + x^4 + \ldots \qquad -1 < x < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad -\infty < x < \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad -\infty < x < \infty$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad -1 < x < 1$$

$$\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \qquad -1 < x < 1$$